Atomare & Subatomare 30.10.08

2) Lagrange Formulierung

Quantenfeldtheorien ableiten // nur Vorbereitungen

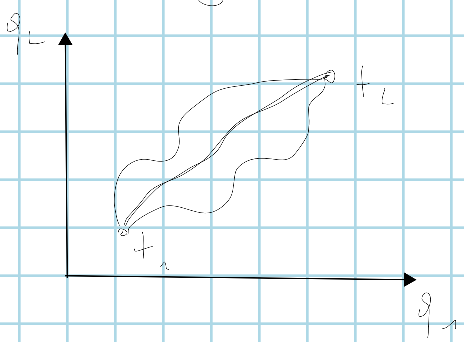
klassisches - nicht dispersives System

$L(q, \dot{q}, t) \longrightarrow L(q, \dot{q})$ // hier nicht expl. zeitabh.

q_i // $i = 1 \dots 3N$

$\frac{\partial L}{\partial t} = \emptyset$ $\qquad L(q, \dot{q}) = \underset{\text{kin}}{T} - \underset{\text{pot}}{V}$

Wirkung $S[q] = \int_{t_1}^{t_2} dt \; L(q, \dot{q})$



$$\delta S[q] = \emptyset$$

$$\boxed{\delta S[q] = \delta \int_{t_1}^{t_2} dt \; L(q, \dot{q}) = \emptyset}$$

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i}_{*_1}$$

// $\delta \dot{q}_i = \delta \frac{d}{dt} q_i = \frac{d}{dt} \delta q_i$

$*_1$: $\frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} (\delta q_i)$

$\longrightarrow \delta L = \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i$

$$\delta S[q] = \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \delta q_i \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot q_i} \right) \delta q_i \right]$$

$$= \int_{t_1}^{t_2} dt \sum \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \right) \right] \delta q_i + \underbrace{\frac{\partial L}{\partial \dot q_i} \delta q_i \Big|_{t_1}^{t_2}}_{\emptyset}$$

$$\delta q_i(t_1) = \delta q_i(t_2) = \emptyset$$

$$\delta S[q] = \int dt \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \right) \right] \delta q_i = \emptyset$$

kann beliebig sein

$\longrightarrow$ vorderer Term

muss $= 0$ sein!

$\Longrightarrow$ Euler Lagrange Gl:

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \right) = \emptyset} \quad \forall\, i$$

$$L = \frac{m\dot x^2}{2} - V \qquad \text{Einsetzen in ELG:}$$

$$m\ddot x = -\frac{d}{dx} V(x) \qquad // \text{ Definition d. Kraft}$$

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot q_i + \frac{\partial L}{\partial \dot q} \ddot q_i \right) = \sum_i \underbrace{\left\{ \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \right) \right] \dot q_i + \frac{\partial L}{\partial \dot q_i} \ddot q_i \right\}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \cdot \dot q_i \right)}$$

$$\& \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \right) = \emptyset$$

$$\longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \dot q_i - L \right) = \emptyset \qquad \frac{\partial L}{\partial \dot q_i} = p_i$$

verallg. Impuls

$$\frac{d}{dt}\left(p_i \dot q_i - L \right) = \emptyset \qquad L(q, \dot q) \longrightarrow H(q, p)$$

(q, p) : Phasenraum

// Legendre Trafo

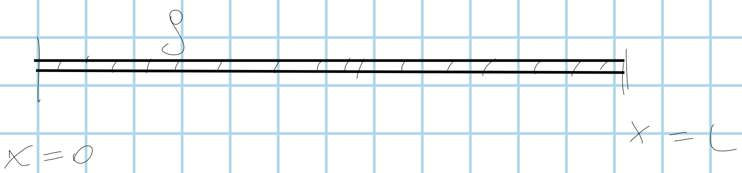
Hamilton Fkt:

$$H = \sum_i p_i \dot{q}_i - L$$

hier: $\frac{dH}{dt} = \emptyset$

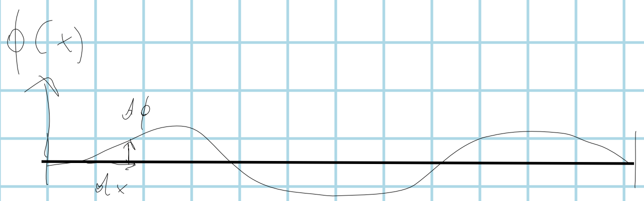
Kontinuierliche Systeme

// Bsp: Feder



ρ : Massenbelegung

F : Spannung



ϕ : Auslenkung aus Ruhelage

// Änderung der Länge berechnen

$$\int_0^l \left(dx^2 + d\phi^2\right)^{\frac{1}{2}} =$$

$$\int_0^l dx \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2}$$

// mit $d\phi = \frac{\partial \phi}{\partial x} dx + \dots$

Änderung der Länge: $\int_0^l dx \; \frac{1}{2} \left(\frac{\partial \phi}{\partial x}\right)^2$

$$\boxed{V = \frac{1}{2} F \int_0^l dx \left(\frac{\partial \phi}{\partial x}\right)^2}$$

$$\boxed{T = \int_0^l dx \; \frac{1}{2} \rho \left(\frac{\partial \phi}{\partial t}\right)^2}$$

$$L = T - V = \int_0^l dx \; \underbrace{\mathcal{L}}_{\text{Lagrange-Dichte}}$$

$$\mathcal{L} = \frac{1}{2} \rho \left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2} F \left(\frac{\partial \phi}{\partial x}\right)^2$$

V : $F \cdot$ Längenänd

$$S = \int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} dt \int_0^l dx\; \mathcal{L}$$

$$\mathcal{L}(\dot\phi, \phi') \qquad \phi' = \frac{\partial \phi}{\partial x}$$

// weder expl. Zeitabh. noch vom Feld – Sonderfall

$$\delta S = 0$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot\phi}\, \delta \dot\phi + \frac{\partial \mathcal{L}}{\partial \phi'}\, \delta \phi'$$

// wieder part. $\int$

$$\delta \dot\phi = \frac{d}{dt}(\delta \phi) \qquad \delta \phi' = \frac{d}{dx}(\delta\phi)$$

$\delta \phi (x,t) = 0$ für folg. Fälle:

$x = l$, $x = 0$, $t = t_1$, $t = t_2$

$$\delta \phi (x, t_1) = 0$$

$$\delta \phi (x = 0, t) = 0 \quad \ldots$$

$$\delta S = \int_{t_1}^{t_2} dt \int_0^l dx \underbrace{\left[\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot\phi} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right)\right]} \delta\phi = 0$$

$$\boxed{\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot\phi} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \phi'} = 0}$$

ELG

BWGL für die gesp. Saite $\qquad$ $\mathcal{L}$ in ELG einsetzen

$$\rho \frac{\partial^2 \phi}{\partial t^2} - F \frac{\partial^2 \phi}{\partial x^2} = 0$$

Impulsdichte definieren $\qquad \pi = \dfrac{\partial \mathcal{L}}{\partial \dot\phi}$

// kann auch vektoriell sein

Hamilton - / Energiedichte

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

$$E = \int dx \, \mathcal{H} = \int dx \left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right]$$

Anschauen: Lorentz invar. Dichte

$$\mathcal{L}(\phi, \partial_\mu \phi)$$ // Verallg. von $L(q, \dot{q})$

$\mu = 0, 1, 2, 3$

$$S[\phi] = \int dt \int dx^1 \int dx^2 \int dx^3 \, \mathcal{L}$$

$$= \int \underbrace{dx^0 dx^1 dx^2 dx^3}_{\text{Lorentzinvar.}} \, \mathcal{L}$$

Aufg. wie Skalar

muss ein Skalar unter Lorentztr. sein!!

$$\mathcal{L} = \frac{1}{2} \left[(\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2 \right]$$ führt zur Klein-Gordon Gl.

// nun immer Masse & Integrat. ändern

γ, Gluonen, $W^\pm$, Z Bosonen

π - Mesonen

$$\delta S = \delta \int d^4x \, \mathcal{L} = \emptyset$$

$$\mathcal{L}(\phi, \partial_\mu \phi)$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \underbrace{\delta (\partial_\mu \phi)}_{\partial_\mu (\delta \phi)}$$

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi = 0$$

$$\boxed{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] = 0}$$ // ELG

$$\mathcal{L} = \frac{1}{2} \left[(\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2 \right]$$
$$= \frac{1}{2} \left[g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - m^2 \phi^2 \right]$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2 g^{\mu\nu} (\partial_\nu \phi) \frac{1}{2} = g^{\mu\nu} (\partial_\nu \phi)$$
$$= \partial^\mu \phi$$

$$g^{\mu\nu} = g^{\nu\mu}$$

$$\rightarrow \boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0} \quad \text{KG-Gl.}$$

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \phi + m^2 \phi = 0$$

Phase

Ansatz: $\boxed{\phi(\vec{r}, t) = a_k \cos(\vec{k}\vec{r} - \omega_k t + \vartheta_k)}$ // ebene Welle

$\omega^2 = k^2 + m^2$ // $\hbar$, usw $= 1$ gesetzt

$(\hbar\omega)^2 = \hbar^2 k^2 c^2 + m^2 c^4$ // rel. Energie-Impuls-Bez.

a_k: Amplitude

Volumen V $\rightarrow$ Wellenzahlen diskret

$$\vec{k} = \left(\frac{2\pi}{L} n_x, \frac{2\pi}{L} n_y, \frac{2\pi}{L} n_z \right)$$

$$\boxed{\phi(\vec{r}, t) = \frac{1}{\sqrt{V}} \sum_{k} \left[\frac{a_{\vec{k}}}{\sqrt{2\omega_{\vec{k}}}} e^{i(\vec{k}\vec{r} - \omega t)} + \frac{a^*_{\vec{k}}}{\sqrt{2\omega_{\vec{k}}}} e^{-i(\vec{k}\vec{r} - \omega t)} \right]}$$

Im Skriptum muss es + sein! → reelles Feld

QFT: $a_{\vec{k}} \longrightarrow \hat{a}_{\vec{k}}$ // werden Erzeugungs-&

$a^*_{\vec{k}} \longrightarrow \hat{a}^+_{\vec{k}}$ // Vernichtungs-Op.

$\mathcal{L}(\phi, \partial_\mu \phi)$ $\quad x^\mu \to x^\mu + d a^\mu$

// einen Schritt weiter gehen! $\quad \delta\phi = (\partial_\mu \phi)\, \delta a^\mu$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi)$$

$$// \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \qquad \text{BWGL}$$

dann einsetzen

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi)$$

$$\frac{\partial \mathcal{L}}{\partial x^\mu} \delta a^\mu = \delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right) d a^\nu$$

$$\delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\mu} \delta a^\mu = \frac{\partial \mathcal{L}}{\partial x^\mu} \delta^\mu_\nu \delta a^\nu$$

$$\boxed{\partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right\} = 0}$$

Energie-Impuls-Tensor $\quad \boxed{T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}}$

$$\boxed{\partial_\mu T^\mu_\nu = 0}$$